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The low frequency sound from multipole sources in axisymmetric shear flows, with applications to jet noise

By M. E. GOLDSTEIN

National Aeronautics and Space Administration, Lewis Research Center, Cleveland, Ohio 44135

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We have obtained a closed-form solution for the sound radiation from multipole sources imbedded in an infinite cylindrical jet with an arbitrary velocity profile. It is valid in the limit where the wavelength is large compared with the jet radius. Simple formulae for the acoustic pressure field due to convected point sources are also obtained. The results show (in a simple way) how the mean flow affects the radiation pattern from the sources. For convected lateral quadrupoles it causes the exponent n of the Doppler factor $(1-M\cos\theta)^{-n}$ multiplying the far-field pressure signal to be increased from the value of 3 used by Lighthill to 5.

1. Introduction

The study of the acoustic radiation from sound sources in a moving flow not only has its own intrinsic value but also has important applications to the understanding of jet noise. For example, Lighthill (1952, 1954) attributed the observed directivity pattern of the sound radiated from subsonic jets to the downstream beaming caused by motion of quadrupole sources through a stationary medium (frequently referred to as the source convection effect). Since then it has been pointed out a number of times (Phillips 1960; Csanady 1966; Mani 1972; Goldstein & Howes 1973) that the mean flow field in the jet can significantly influence this convection effect. Csanady (1966) indicated that at sufficiently high frequencies the mean flow surrounding the source should cause the effect to disappear. Moreover it is now recognized that in the low frequency limit the surrounding flow will act to augment the convective amplification (Doppler) factor found by Lighthill (1952, 1954). Up to now results have been obtained only for the case of a slug-flow velocity profile (Slutsky & Tamagno 1961; Graham & Graham 1971; Mani 1972). The purpose of the present report is therefore to determine the effect of an arbitrary radial velocity profile on the low frequency (wavelength long compared with the transverse dimension of the jet) acoustic radiation due to multipole sources (monopole, dipole and quadrupole) embedded in an infinite cylindrical jet (see figure 1).

In this limit it is possible to obtain a relatively simple closed-form solution which shows that the mean flow does indeed influence the convection effect and that this influence can be described in a very simple way. It also shows that the far-field acoustic radiation from point monopole and axial dipole sources depends

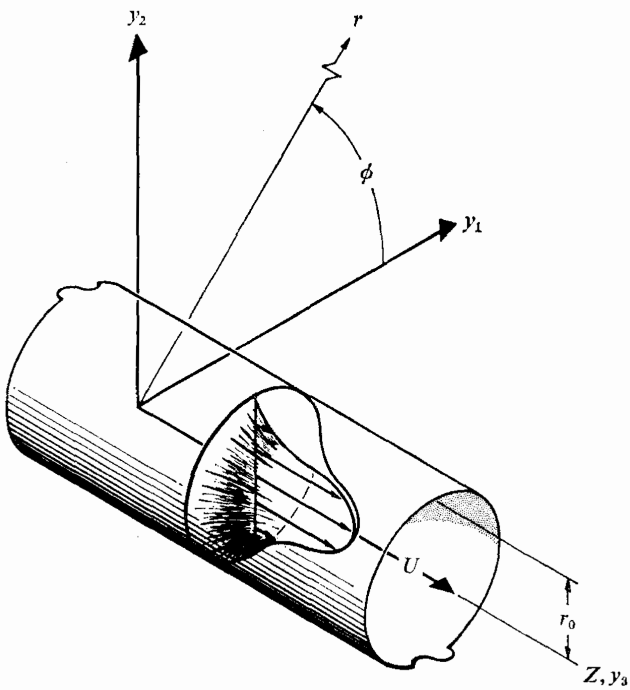

FIGURE 1. Dimensionless co-ordinate system for jet flow.

only on the local value of the mean flow velocity while the radiation from *lateral* quadrupole sources is very strongly influenced by the local velocity gradient. The exponent n of the Doppler factor $(1-M\cos\theta)^{-n}$ multiplying the far-field pressure from this quadrupole is increased from the value of 3 used by Lighthill (1954) for sources moving through a medium at rest to 5.

The present solution leads to an explanation for certain experimental observations relating to low frequency jet noise.

2. Analysis

The linearized equations governing the propagation of sound in an axisymmetric transversely sheared mean flow (see figure 1) with mean density ρ_0 and sound speed a_0 are (see Goldstein 1974, p. 10)

$$U_s\rho_0 D_0 u_r/D\tau = -\partial p/\partial r + r_0 f_r, \tag{1}$$

$$\nabla^2 p - M^2 \frac{D_0^2 p}{D\tau^2} + 2\rho_0 U_s U' \frac{\partial u_r}{\partial Z} = r_0\left(\nabla . \mathbf{f} - a_0\rho_0 M \frac{D_0 q}{D\tau}\right), \tag{2}$$

where $D_0/D\tau \equiv \partial/\partial\tau + U\,\partial/\partial Z$, p is the pressure fluctuation, u_r is the radial component of the acoustic particle velocity and U_s denotes a convenient reference value of the mean velocity $U_s U(r)$ in the axial (Z) direction. All lengths have been made dimensionless using a characteristic radius r_0 of the jet, and the time τ has been non-dimensionalized using U_s/r_0. The constant $M = U_s/a_0$ is a characteristic Mach number of the mean flow. The prime denotes differentiation with respect to the dimensionless radial co-ordinate r. We assume that the flow contains a volume source of mass $\rho_0 q$ and an externally applied force per unit volume $\mathbf{f}$. Then q can be thought of as the strength of an arbitrary volume-monopole source and $\mathbf{f}$ can be thought of as the strength of an arbitrary volume-dipole source.† We can introduce a volume-quadrupole source (of strength T_{ij}) by replacing f_i by

$$f_i = D_i + r_0^{-1}\,\partial T_{ij}/\partial y_j, \tag{3}$$

where $\{y_j\}$ denotes dimensionless rectangular Cartesian co-ordinates and $\mathbf{D}$ now denotes the dipole strength. As is usual in the analysis of problems concerning sound propagation in sheared flows u_r is eliminated between (1) and (2) to obtain the third-order wave equation‡

$$\frac{D_0}{D\tau}\left(\nabla^2 p - M^2\frac{D_0^2 p}{D\tau^2}\right) - 2U'\frac{\partial^2 p}{\partial Z\,\partial r} = r_0\left[\frac{D_0}{D\tau}\nabla\,.\,\mathbf{f} - 2U'\frac{\partial f_r}{\partial Z} - \rho_0 a_0 M\frac{D_0^2 q}{D\tau^2}\right]. \tag{4}$$

The variables Z and τ can be eliminated from this equation in the usual way by introducing the two-dimensional Fourier transform

$$p = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(r,\phi\,|\,k,\epsilon)\, e^{-i(\epsilon\tau + kZ)}\,dk\,d\epsilon \tag{5}$$

to obtain

$$\begin{aligned}\nabla_t^2 P - \frac{2\kappa U'}{1+\kappa U}\frac{\partial P}{\partial r} + \epsilon^2[M^2(1+\kappa U)^2 - \kappa^2]P \\ = \frac{i r_0}{(2\pi)^2\epsilon(1+\kappa U)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\epsilon(\kappa Z+\tau)}\left[\frac{D_0}{D\tau}\nabla\,.\,\mathbf{f} - 2U'\,\frac{\partial f_r}{\partial Z} - \rho_0 a_0 M\frac{D_0^2 q}{D\tau^2}\right]dZ\,d\tau \\ = \frac{r_0(1+\kappa U)^2}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\epsilon(\kappa Z+\tau)}\left\{\nabla\,.\left[\frac{\mathbf{f}}{(1+\kappa U)^2}\right] + \frac{i\rho_0 a_0 M\epsilon q}{1+\kappa U}\right\}dZ\,d\tau,\end{aligned} \tag{6}$$

where $\nabla_t^2 = \nabla^2 - \partial^2/\partial Z^2$ denotes the transverse part of the Laplace operator and $\kappa \equiv k/\epsilon$.

We are interested in obtaining a low frequency solution to this equation. Thus suppose that the source distributions $\mathbf{f}$ and q are such that their Fourier transforms vanish unless the dimensionless frequency (Strouhal number)

$$\epsilon \equiv \omega r_0/U_s \tag{7}$$

is close to zero and the wavenumber ratio $\kappa = k/\epsilon$ is of order one. In order to simplify the analysis assume that

$$\mathbf{f} = (0, 0, f_3) \tag{8}$$

has a component only in the axial (mean flow) direction. Then in the neighbourhood of the jet where r is of order one (i.e. the inner region) we seek an expansion of the form $\quad P = \epsilon\ln\epsilon P^{(-1)} + \epsilon P^{(0)} + \alpha(\epsilon)P^{(1)} + \ldots\ \alpha(\epsilon) = o(\epsilon)$

† These definitions are, for purposes of comparison, taken to be consistent with those given in Morse & Ingard (1968, § 11.2) for sources moving through a stationary medium.

‡ When T_{ij} is put equal to minus the turbulent Reynolds stress and q and D_i are put equal to zero, (4) with f_i given by (3) describes the noise generated by a turbulent flow. In fact, Goldstein & Howes (1973; see also Goldstein 1974, p. 394) showed that this equation can be obtained by combining Phillips' (1960) equation with the momentum equation after neglecting the products of acoustic quantities both with themselves and with the turbulent quantities.

to obtain in the limit as $\epsilon \to 0$ with r held fixed

$$\nabla_t^2 P^{(-1)} - \frac{2\kappa U'}{1+\kappa U}\frac{\partial P^{(-1)}}{\partial r} = 0,$$

$$\nabla_t^2 P^{(0)} - \frac{2\kappa}{1+\kappa U} U' \frac{\partial P^{(0)}}{\partial r}$$

$$= -\frac{i(1+\kappa U)^2 r_0}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\epsilon(\kappa Z+\tau)}\left[\frac{\kappa f_3}{(1+\kappa U)^2} - \frac{\rho_0 a_0 M q}{1+\kappa U}\right] dZ\, d\tau.$$

If we expand the solution to the latter equation in the Fourier series

$$P^{(0)} = \sum_{n=-\infty}^{\infty} P_n^{(0)}(r)\, e^{in\phi}$$

we find that the lowest Fourier coefficient is determined by

$$\frac{d}{dr}\left(\frac{r}{(1+\kappa U)^2}\frac{dP_0^{(0)}}{dr}\right) = -\frac{irr_0}{(2\pi)^3}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\epsilon(\kappa Z+\tau)}\left[\frac{\kappa f_3}{(1+\kappa U)^2} - \frac{\rho_0 a_0 M q}{1+\kappa U}\right] d\phi\, dZ\, d\tau$$

and as a result

$$P_0^{(0)} = -\frac{ir_0}{(2\pi)^2}\int_{\infty}^{r}\frac{(1+\kappa U)^2}{r}\left\{\int_0^r\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\epsilon(\kappa Z+\tau)}\right.$$

$$\left.\times\left[\frac{\kappa f_3}{(1+\kappa U)^2} - \frac{\rho_0 a_0 M q}{1+\kappa U}\right] r\, dr\, d\phi\, dZ\, d\tau\right\} dr + \text{constant}. \quad (9)$$

Hence, assuming that the source distributions decay sufficiently rapidly with r as $r \to \infty$,

$$P_0^{(0)} \sim b_0 + c_0 \ln r,$$

where
$$c_0 \equiv -\frac{ir_0}{(2\pi)^3}\int_{-\infty}^{\infty}\int e^{i\epsilon(\kappa Z+\tau)}\left[\frac{\kappa f_3}{(1+\kappa U)^2} - \frac{\rho_0 a_0 M q}{1+\kappa U}\right] d\mathbf{y}\, d\tau,$$

in which the omission of the limits on the second integral denotes an integration over all space and $d\mathbf{y}$ denotes a (three-dimensional) volume element. Upon explicitly introducing the quadrupole source through (3) this becomes after integration of the quadrupole term by parts

$$c_0 = -\frac{ir_0}{(2\pi)^3}\int_{-\infty}^{\infty}\int e^{i\epsilon(\kappa Z+\tau)}\left[\frac{2\kappa^2 U' T_{3r}}{r_0(1+\kappa U)^3} + \frac{\kappa D_3}{(1+\kappa U)^2} - \frac{\rho_0 a_0 M q}{1+\kappa U}\right] d\mathbf{y}\, d\tau, \quad (10)$$

where
$$T_{3r} \equiv T_{3j}\frac{\partial r}{\partial y_j} = \frac{y_1}{r}T_{31} + \frac{y_2}{r}T_{32}.$$

Similarly, if the source distributions vanish faster than any power of r as $r \to \infty$, it can be shown that

$$P_n^{(0)} \sim b_n + c_n^{\pm} r^{\pm n} \quad \text{as} \quad r \to \infty, \quad n \neq 0. \quad (11)$$

This solution obviously breaks down at large distances from the jet.

In order to determine the properties of the sound in the radiation field we must construct an outer expansion. To this end we proceed in the usual way by introducing the outer variable

$$\tilde{r} = \epsilon r \quad (12)$$

into (6) and expanding its solution for $\epsilon \to 0$ with $\tilde{r}$ held fixed. If it is again assumed that both the mean velocity field of the jet and the source distribution vanish sufficiently fast as $r \to \infty$, equation (6) will reduce to the ordinary wave equation†

$$\tilde{\nabla}_t^2 P + (M^2 - \kappa^2) P = 0,$$

where $\tilde{\nabla}_t^2$ is the Laplacian $\epsilon^{-2}\nabla_t^2$ in the outer variables. Then the outer expansion is

$$P = \epsilon \tilde{P}^{(0)}(\tilde{r}, \phi) + \beta(\epsilon) \tilde{P}^{(1)}(\tilde{r}, \phi) + \ldots \beta(\epsilon) = o(\epsilon) \tag{13}$$

and at least for the first few m the $\tilde{P}^{(m)}$ are determined by

$$\tilde{\nabla}^2 \tilde{P}^{(m)} + K^2 \tilde{P}^{(m)} = 0,$$

where

$$K = (M^2 - \kappa^2)^{\frac{1}{2}}. \tag{14}$$

This equation will possess an outgoing-wave solution only if K is real, in which case the solution can be written as

$$\tilde{P}^{(0)} = \sum_{n=-\infty}^{\infty} C_n H_n^{(1)}(K\tilde{r})\, e^{in\phi}, \tag{15}$$

where $H_n^{(1)}$ denotes a Hankel function.

The constants b_n, c_n and C_n in the inner and outer expansions are determined by requiring the expansions to 'match' to the proper order in some intermediate region. This can be accomplished either by using the matching principle of Van Dyke (1964, p. 64) or by introducing intermediate variables and re-expanding in the overlap domain (Cole 1968, p. 234). In either case we find that

$$\left.\begin{aligned} P^{(-1)} &= \text{constant}, \\ C_0 = -\tfrac{1}{2}\pi i c_0, \quad C_n &= 0, \quad n = \pm 1, \pm 2, \ldots \end{aligned}\right\} \tag{16}$$

Hence, it follows from (5), (10), (12) and (14)–(16) that to lowest order in ϵ the pressure fluctuation in the outer region is given by

$$p \sim \frac{r_0}{8\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \epsilon \exp\{-i\epsilon\{\kappa[Z - Z_0(\tau)] + [\tau + \kappa Z_0(\tau)]\}\} H_0^{(1)}(\epsilon K r)\, \mathscr{P}(k, \epsilon)\, dk\, d\epsilon, \tag{17}$$

where

$$\mathscr{P}(k, \epsilon) \equiv -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int e^{i\epsilon(\kappa Z + \tau)} \left[\frac{2\kappa^2 U' T_{3r}}{r_0 (1 + \kappa U)^3} + \frac{\kappa D_3}{(1 + \kappa U)^2} - \frac{\rho_0 a_0 M q}{1 + \kappa U} \right] d\mathbf{y}\, d\tau \tag{18}$$

and $Z_0(\tau)$ has been added and subtracted in the exponent for future convenience. (If the source region is moving Z_0 can be thought of as the axial co-ordinate of this region at some appropriate time; if the source region is stationary no generality is lost if Z_0 is set equal to zero.) When r is much larger than the wavelength we can replace $H_0^{(1)}$ in (17) by its asymptotic expansion to obtain

$$p \sim \frac{r_0}{8\pi} \left(\frac{2}{i\pi r} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\{i\epsilon[\kappa(Z - Z_0) - Kr]\}}{(K\epsilon)^{\frac{1}{2}}} \epsilon \exp[-i\epsilon(\tau + \kappa Z_0)]\, \mathscr{P}(k, \epsilon)\, dk\, d\epsilon.$$

Upon introducing the polar co-ordinates

$$R \equiv \{r^2 + (Z - Z_0(\tau))^2\}^{\frac{1}{2}}, \quad \theta \equiv \tan^{-1}\{r/[Z - Z_0(\tau)]\}$$

† At least to any order of ϵ which is of interest.

this can be written as

$$p \sim \frac{r_0}{8\pi}\left(\frac{2}{i\pi R\sin\theta}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} \epsilon^{\frac{1}{2}} e^{-i\epsilon\tau}\,\Gamma(\epsilon)\,d\epsilon, \tag{19}$$

where $$\Gamma(\epsilon) \equiv \int_{-\infty}^{\infty} \frac{\exp\left[-i\epsilon R(\kappa\cos\theta - K\sin\theta) - i\epsilon\kappa Z_0\right]}{K^{\frac{1}{2}}}\,\mathscr{P}(k,\epsilon)\,dk.$$

But for $\epsilon R \gg 1$ we can use the method of stationary phase (Erdélyi 1956, equation (2), §2.9) to obtain the following asymptotic expansion for Γ:

$$\Gamma(\epsilon) \sim \left[\frac{2\epsilon\pi\sin(\theta)}{iR}\right]^{\frac{1}{2}} \mathscr{P}(-\epsilon M\cos\theta, \epsilon)\exp\left[i\epsilon(Z_0 M\cos\theta + MR)\right] \quad \text{as} \quad \epsilon R \to \infty.$$

Hence

$$p \sim \frac{r_0}{4\pi i R}\int_{-\infty}^{\infty} \epsilon\exp\{i\epsilon[MR - (\tau - MZ_0\cos\theta)]\}\,\mathscr{P}(-\epsilon M\cos\theta,\epsilon)\,d\epsilon, \tag{20}$$

where

$$\mathscr{P}(-\epsilon M\cos\theta,\epsilon) = -\frac{M}{2\pi}\int_{-\infty}^{\infty}\int \exp\left[i\epsilon(\tau - ZM\cos\theta)\right]\left[\frac{2MU'\cos^2\theta T_{3r}}{r_0(1-MU\cos\theta)^3} - \frac{\cos\theta D_3}{(1-MU\cos\theta)^2} - \frac{\rho_0 a_0 q}{1-MU\cos\theta}\right] dy\,d\tau. \tag{21}$$

This is the final formula for the acoustic pressure fluctuations in the radiation field due to monopole, dipole and quadrupole sources of strengths $\rho_0 q$, D_3 and T_{3r} respectively. In order to understand its significance it is useful to consider the special case of point sources carried along at the local velocity of the flow.

Convected point sources

Consider a harmonic point source with dimensionless frequency ϵ_0 moving with a convection velocity U_c (which may in general be different from the local jet velocity $U(r_s)U_s$ at the dimensionless source radius r_s). For convenience we take the normalizing velocity U_s to be the source convection velocity U_c. In order to emphasize this choice we shall write M_c in place of $M \equiv U_s/a_0$. Then the source strengths must be of the form

$$\begin{pmatrix}\rho_0 a_0 q\\ D_3\\ T_{3r}\end{pmatrix} = \frac{1}{r_0^3}\exp(-i\epsilon_0\tau)\,\delta(Z-\tau)\frac{\delta(r-r_s)}{2\pi r}\delta(\phi)\begin{pmatrix}Q_M\\ Q_D\\ Q_Q\end{pmatrix}. \tag{22}$$

For purposes of comparison we take $Z_0(\tau)$ to be the dimensionless position of the source at the emission time $r_0(\tau/U_c - R/a_0)$ of the sound wave which reaches the point $(r_0 r, \phi, r_0 Z)$ at the time $r_0\tau/U_c$ (see figure 2). Thus

$$Z_0 = \tau - M_c R. \tag{23}$$

Then, inserting (22) into (21) shows that

$$\mathscr{P}(-\epsilon M_c\cos\theta,\epsilon) = \frac{M_c}{r_0^3}\delta(\epsilon_0 - \epsilon(1-M_c\cos\theta))\left[\frac{Q_M}{1-M_j\cos\theta} + \frac{\cos\theta\, Q_D}{(1-M_j\cos\theta)^2} - \frac{2M_j'\cos^2\theta\, Q_Q}{r_0(1-M_j\cos\theta)^3}\right], \tag{24}$$

where $M_j \equiv U(r_0)M_c$ is the local jet Mach number at the radial location of the source and $M_j' \equiv M_c U'(r_s)$ is the Mach number gradient at the source location.

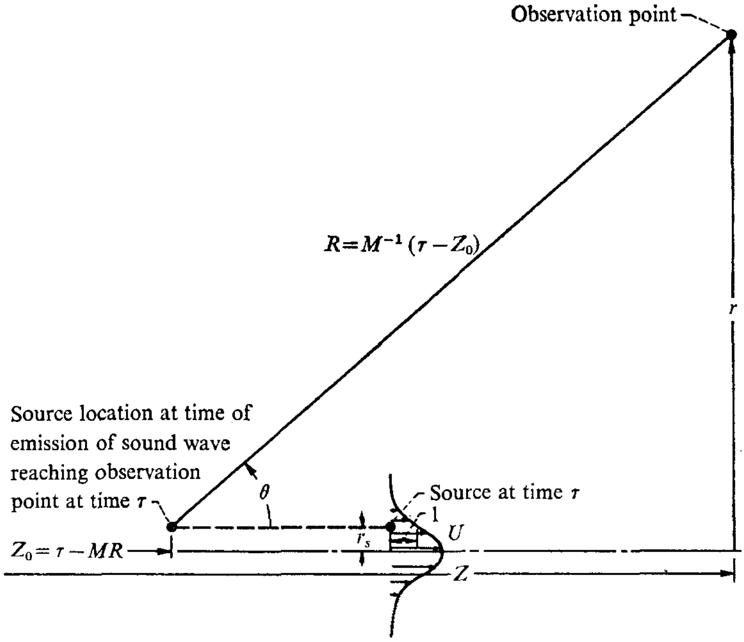

FIGURE 2. Co-ordinate system for convected point source.

Inserting this together with (23) into (20) now shows that the pressure fluctuations at sufficiently large values of the distance $r_0 R$ between the observation point and the source point at the time of emission are given by

$$p \sim \frac{M_c \epsilon_0/r_0}{4\pi i(r_0 R)(1-M_c\cos\theta)^2}\left[\frac{Q_M}{1-M_j\cos\theta}+\frac{\cos(\theta)\,Q_D}{(1-M_j\cos\theta)^2}-\frac{2M_j'\cos^2(\theta)\,Q_Q}{r_0(1-M_j\cos\theta)^3}\right] \times \exp\left[i\epsilon_0(M_c R-\tau)\right]. \quad (25)$$

It is instructive to compare this result with the equivalent results for point monopoles and axial dipoles moving with a Mach number M_c through a stationary medium which are given in Morse & Ingard (1968, §11.2). In terms of the present notation they are, respectively,

$$p_M \sim \frac{(M_c\epsilon_0/r_0)\,Q_M \exp\left[i\epsilon_0(M_c R-\tau)\right]}{4\pi i(Rr_0)(1-M_c\cos\theta)^2} \quad \text{as} \quad R\to\infty,$$

$$p_{D\,\mathrm{axial}} \sim \frac{(M_c\epsilon_0/r_0)\cos(\theta)\,Q_D \exp\left[i\epsilon_0(M_c R-\tau)\right]}{4\pi i(Rr_0)(1-M_c\cos\theta)^2} \quad \text{as} \quad R\to\infty.$$

Notice that these formulae reduce to corresponding terms in (25) when $M_j\to 0$ with M_c held fixed. The latter equation shows that there is a contribution $(1-M_c\cos\theta)^{-2}$ to the Doppler factor due to source convection with the remaining contribution coming from the mean flow velocity.

Now consider the case when M_j and M_c are equal. Then these results show that the net effect of embedding a moving monopole source in a narrow region (relative to the wavelength) of moving flow is to change the exponent of its convection factor $1-M\cos\theta$ from -2 to -3 while the net effect on a dipole source is to change the exponent of its convection factor from -2 to -4.

Equation (25) shows that not only is the acoustic power altered but the mean flow causes an additional beaming of the sound in the direction of motion with a dipole source being more strongly affected than a monopole. Moreover, this effect is substantially independent of the detailed shape of the velocity profile and depends only on the local velocity at the source.

Although the definitions of the strengths of dipole and quadrupole sources are pretty standard, there is some variation in what is called the strength of a monopole source: in a stationary medium some authors consider $\partial q/\partial\tau$ rather than q to be the source strength.† Then for a moving medium the corresponding definition of the source strength would be $D_0 q/D\tau$. It is not hard to show that in the latter case the convection factor remains unaltered with this new definition while in the former (stationary medium) case the exponent of the convection factor is reduced to -1. Then the ratio of the pressure for a monopole source in a stationary medium to that in a moving medium will be the same as for the corresponding dipole sources.

The results for the lateral (Z, r) quadrupole source are even more remarkable. First, the convection factor for a point quadrupole source moving through a stationary medium (Lighthill 1952) is $(1-M\cos\theta)^{-3}$. The present equations show that the effect of the mean flow on the convection factor is essentially the same as it is for a dipole source (i.e. it alters the exponent by 2). More important, however, they also show that there are at least some components of the quadrupole source which are proportional to the frequency ϵ_0. This is in contrast to the case of a stationary medium, when all components are proportional to ϵ_0^2. Thus in the limit of very low frequency the mean flow acts to keep the efficiency of certain components of the quadrupole source at the same level as that of a dipole source. In a stationary medium the quadrupole is always a much less efficient emitter of sound than the dipole at low frequencies. Although Lighthill (1954) originally used these words for an entirely different reason we may interpret this result as an indication that "the velocity gradient acts like a sounding board to increase the efficiency of the quadrupole radiation (to that of a dipole)".

The proportionality to the Mach number gradient indicates that (at a fixed Strouhal number) this component of the quadrupole source will become progressively more important as the jet Mach number increases. Moreover, this proportionality also shows that this quadrupole source will not appear if the local velocity gradient is zero as it is for a top-hat velocity profile. However, it should be pointed out that the present analysis does not rule out the existence of other components of the quadrupole which do not have one of their axes aligned with the Z direction. In fact an examination of the equations shows that it is highly likely that the longitudinal $(r, -r)$ quadrupole will also make a contribution.‡

When we use the alternative definition of the monopole source strength given

† The source will then not correspond to a source of volume flow.

‡ In fact it is entirely possible that this quadrupole will also have its Doppler factor augmented by the mean flow. Indeed one of the reviewers has pointed out that Mani (1974) has shown that a quadrupole source in a plug-flow jet has the exponent of its Doppler factor augmented by 2. The present results would indicate that the source found by Mani ought to be a longitudinal quadrupole.

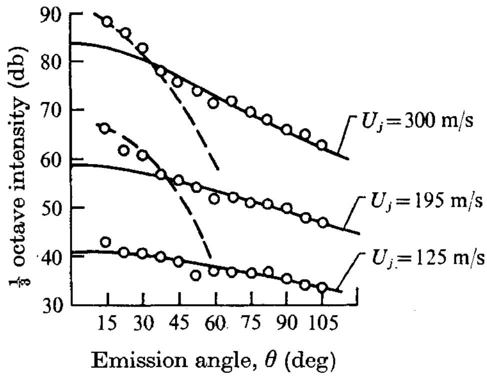

FIGURE 3. Directivity of $\frac{1}{3}$ octave intensity for $U_s \sigma_0 / U_j(1 - M\cos\theta) = 0{\cdot}03$. ○, data from Lush (1971); ——, Lighthill's directivity factor $(1 - M\cos\theta)^{-5}$; – – –, $\cos^4\theta/(1 - M\cos\theta)^9$. U_j = jet velocity; $U_s/U_j = 0{\cdot}65$.

above we can say that the present results show that for all the sources in (25) the mean flow always acts to introduce the additional Doppler factor $(1 - M\cos\theta)^{-2}$ multiplying the pressure.

Implications for low frequency jet noise

Lighthill (1952, 1954) attributed the directivity of jet noise to the large exponent of the quadrupole convection factor. Since the pressure of a point quadrupole is proportional to $(1 - M\cos\theta)^{-3}$ he deduced that the acoustic intensity should have a directivity factor of $(1 - M\cos\theta)^{-6}$. But Ffowcs Williams (1960) showed that owing to the effects of finite volume this directivity factor should actually be $(1 - M\cos\theta)^{-5}$. The present results show that the mean flow can increase the absolute magnitude of the exponent† of the pressure convection factor by 2. Hence the absolute magnitude of the exponent for the acoustic intensity can be increased by 4.

Thus at low frequencies there should be a lateral-quadrupole contribution to the jet noise which is proportional to

$$\cos^4(\theta)\,|M_j'|^2/(1 - M\cos\theta)^9. \tag{26}$$

Owing to the large exponent this term will tend to dominate near‡ the $\theta = 0$ axis and probably be swamped by the other components of the quadrupole at $\theta = 90°$. This could account for the observed concentration of low frequency sound on the jet axis. Lush (1971) measured directivity patterns in $\frac{1}{3}$ octave bands for a subsonic jet. His data for the lowest frequency band are shown in figure 3. The solid curve shows Lighthill's directivity factor (as plotted by Lush). The dashed curves are plots of (26) with the level adjusted to go through the last ($\theta = 15°$) point. Notice that the contribution from the dashed curve is more

† One of the referees has indicated that this exponent has recently been found (although not explicitly pointed out) by Berman (1974).

‡ It is not clear whether this result actually applies on the axis since the method of stationary phase (which was used to obtain the far-field pressures) is not necessarily uniformly valid in θ and could break down at this point.

important at the higher Mach numbers. This is consistent with the $|M_j|^2$ factor which multiplies (26).

Thus it is plausible that the alteration of the quadrupole source by the mean flow could account for the observed concentration of the low frequency sound on the jet axis. This explanation has certain similarities to and differences from the one given by Ribner & MacGregor (1968). They argued that there is a self-noise term (turbulence/turbulence interaction) which is independent of direction and a shear-noise term (turbulence/mean-shear interaction) which is beamed downstream owing to a factor $\cos^2\theta$ which multiplies it. They further argued that the shear noise is of lower frequency than the self-noise, causing a net concentration of the low frequency sound on the axis. The present explanation is similar to theirs in that the quadrupole source (26) is multiplied by the mean shear in equation (25) for the pressure field and therefore is the result of an interaction between the shear and the sound sources (i.e. a shear-noise term). However, our explanation does not imply that the shear-noise spectrum has a lower characteristic frequency than the self-noise spectrum. It merely implies that the *low frequency* part of the noise is concentrated on the jet axis owing to a stronger Doppler factor.

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